THE KEYSTONE OF THE COLLECTION

The Fixed *Point*

Lawvere's theorem — and the single purchase of power and limit. A system rich enough to name itself finds, in the mirror, a point equal to its own image; that one point is the engine of everything it can compute and the source of everything it cannot complete.

THE CONSTRUCTION

$$g(x) = \alpha(\varphi(x)(x))$$

feed a thing to the map it names, then twist · the fixed point is `s = φ(a)(a) = α(s)`

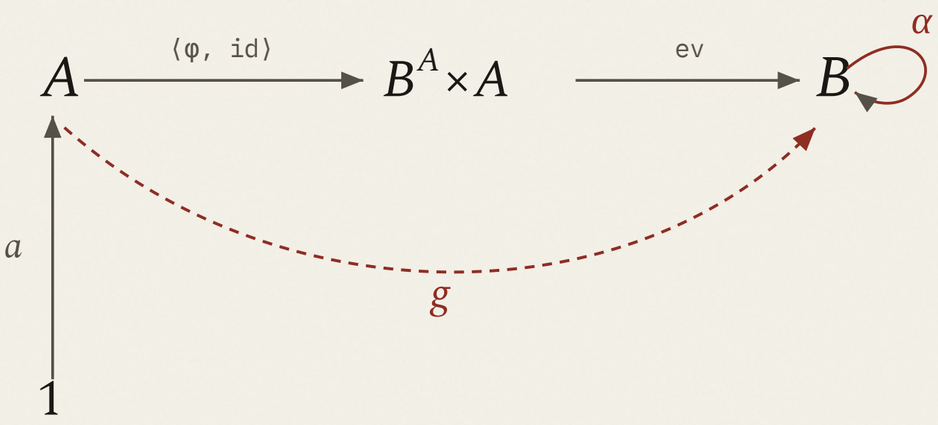
LAWVERE · 1969

> **In a cartesian closed category, if there is a point-surjective map `φ : A → B^A`, then every endomorphism `α : B → B` has a fixed point** — a point `s` with `α(s) = s`.

THE PROOF, IN FIVE LINES

1. Fix `α : B → B`. Form `g = α∘ev∘⟨φ,id⟩`, so *g(a) = α(φ(a)(a)).*
2. Curry `g` to a point `⌜g⌝ : 1 → B^A`.
3. `φ` is point-surjective, so some `a` names `g`: `φ(a) = g`.
4. Set `s = φ(a)(a)`.
5. `s = φ(a)(a) = g(a) = α(φ(a)(a)) = α(s).` ∎

THE DIAGONAL, DRAWN



`φ(a) = g ⟹ s = φ(a)(a) = g(a) = α(s)`

The fixed point s is the value the self-name a takes on itself. The whole content is the diagonal.

Limit

`α fixed-point-free ⟹ no mirror`

If a twist on the answers has no fixed point of its own, the construction demands an object that cannot exist — so the self-naming φ cannot exist. The wall every formal system meets.

Cantor (no A ↠ 2^A) · the **Liar** & Tarski · **Gödel**'s incompleteness · the **halting** problem

Power

`reflexive object (φ = id) ⟹ every α fixed`

Where the answers admit a fixed point — a reflexive object, naming as application — the construction hands one back, uniformly. The engine of all computation.

the **Y-combinator** · **Kleene**'s recursion theorem · **quines** · the **diagonal lemma**

Power and limit are not opposites. They are the two things you cannot have apart.

ISSUED FROM QUODLIBET — FIVE STUDIES IN SELF-REFERENCE

I *The Diagonal*, the construction is one · II *The Atlas*, the territory · III *The Ascent*, no roof · IV *The Descent*, no floor · V *The Mirror*, the keystone

COLOPHON

This plate states the theorem all five volumes rest on, with its proof. The proof is Lawvere's; the reading of power and limit as a single purchase is the collection's synthesis.

Lawvere 1969 · Cantor 1891 · Gödel 1931 · Turing 1936 · Kleene 1938 · Curry & Church · Löb 1955 · Löwenheim–Skolem 1915 / 1922