QUODLIBET

A FOLIO

One *Construction*

self-reference, and the single root of power and limit

$$g(x) = \alpha(\varphi(x)(x))$$

THE PRINT COMPANION TO THE SIX VOLUMES

THE QUESTION

The oldest question logic asks is not about the world but about itself. What happens when a system grows rich enough to describe its own workings — to refer, from inside, to what it can and cannot do?

The question sounds like a curiosity. Its answer is the deepest fact known about the reach of formal reasoning. A system that can name itself acquires a single object — a fixed point — and that one object wears two faces.

One face is everything such a system *can* do: every program, every recursion, every machine that builds another. The other is everything it *cannot*: the truths it cannot prove, the questions it cannot decide, the models it cannot pin down. This folio follows the one construction that is both faces at once.

THE CONSTRUCTION

$$g(x) = \alpha(\varphi(x)(x))$$

Three moves. Let φ give every map a name, so that to name a map and to hold an element of the system become the same act. Apply a map to its own name — the diagonal, the act of self-reference. Then twist the result by some operation α.

The object g so built has a strange property. It cannot be named inside the system without naming, at the same stroke, a *fixed point* of α — a point the twist leaves unmoved. Everything that follows turns on a single question: whether such a point can exist.

THE THEOREM · LAWVERE, 1969

In a setting closed under naming and application, if some φ names every map A → B, then every twist α : B → B has a fixed point.

1. Given α, set g(x) = α(φ(x)(x)).

2. Since φ names every map, some a has φ(a) = g.

3. Then s = φ(a)(a) = g(a) = α(φ(a)(a)) = α(s).

So α(s) = s. ∎

No negation. No numbers. No infinity. The whole of it is: *name, apply, twist.*

THE FACE THAT BOUNDS

Give the twist no fixed point — let α be ¬, the swap of true and false, which agrees with nothing — and the theorem runs in reverse: no map can name them all. There is always one it misses, and the one it misses is the diagonal itself —

$$\{\, x : x \notin \varphi(x) \,\}$$

That missing element is Cantor's uncountable subset. It is Russell's set, the one no consistent theory may hold. It is Gödel's true-but-unprovable sentence, Tarski's undefinable truth, Turing's undecidable halt. One construction, one twist with no fixed point — and every wall that formal reasoning meets.

THE FACE THAT BUILDS

Give the twist a fixed point instead — let the answers form a place where every map already has a name, where to apply a thing to itself is always defined — and the same construction does not forbid the fixed point. It hands it over. And what it hands over is recursion itself —

$$Y\ F = F(Y\ F)$$

the program that calls itself with no name for itself, the function defined in terms of the very thing it defines, the machine that reads its own description and builds a copy. The diagonal that bounded, read forwards, is the diagonal that *builds.*

ONE PURCHASE

The limit and the power are not two facts. They are one. The construction is identical; only the twist differs, and the difference is a single bit — whether α has a fixed point. Where it does not, self-reference is a wall. Where it does, self-reference is an engine. You cannot buy the one without the other: the act that lets a system speak of itself is the act that bounds what it can say.

Power and limit are not opposites.
They are the two things you cannot have apart.

colophon

One Construction is the print distillation of Quodlibet — a six-volume study of self-reference and the single theorem beneath it. Where the volumes chart the territory, this folio keeps only the spine: that one construction is the reason a formal system can compute anything, and the reason it cannot decide everything — and that these are one reason, not two.

I The Diagonal · II The Atlas · III The Engine
IV The Ascent · V The Descent · VI The Mirror
— and the apparatus, Scholia

$$g(x) = \alpha(\varphi(x)(x))$$

After Lawvere, 1969 · Cantor · Gödel · Turing · Kleene

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